Problem 1 (6 points)

(Adapted from Kraus Ch 8)
A radio source has flux densities of $S_1 = 12.1$ Jy and $S_2 = 8.3$ Jy at frequencies of $\nu_1 = 600$ MHz and $\nu_2 = 1415$ MHz, respectively.

A) Show that its spectral index $\alpha = [\log (S_1/S_2)]/[\log (\nu_2/\nu_1)]$ (2 points)

Spectral index $\alpha$; $S_\nu \propto \nu^{-\alpha}$

$$\frac{S_1}{S_2} = \left(\frac{\nu_1}{\nu_2}\right)^{-\alpha} = \left(\frac{\nu_2}{\nu_1}\right)^{\alpha}$$

$$\log \left(\frac{S_1}{S_2}\right) = \alpha \log \left(\frac{\nu_2}{\nu_1}\right)$$

$$\alpha = \log \left(\frac{S_1}{S_2}\right) / \log \left(\frac{\nu_2}{\nu_1}\right)$$

B) Calculate its spectral index. (2 points)

$$\alpha = \log \left(\frac{12.1}{8.3}\right) / \log \left(\frac{1415}{600}\right) = 0.44$$

C) Is the spectrum thermal or nonthermal? (2 points)

Thermal spectra have $\alpha = -2$, so this spectrum is nonthermal.

Problem 2 (8 points)

(Rybicki & Lightman Problem 1.5)
A supernova remnant has an angular diameter $\theta = 4.3$ arc minutes and a flux at 100 MHz of $F_{100} = 1.6 \times 10^{-19}$ erg cm$^{-2}$ s$^{-1}$ Hz$^{-1}$. Assume that the emission is thermal.

A) What is the brightness temperature $T_b$? What energy regime of the blackbody curve does this correspond to? (2 points)

Definition of brightness temperature: $T_b = \frac{Lc^2}{4\pi\epsilon_0}$
To calculate $I_\nu$, need to know the flux density and angular size:

\[
I_\nu = \frac{F_\nu}{\Delta \Omega} \\
\Delta \Omega = \frac{\pi \theta^2}{4} = \frac{\pi}{4} \left( \frac{4.3' \pi}{60 + 180} \right)^2 = 1.2 \times 10^{-6} \text{ Sr} \\
I_\nu = \frac{1.6 \times 10^{-19} \text{ erg cm}^{-2} \text{s}^{-1} \text{Hz}}{1.2 \times 10^{-6} \text{Sr}} = 1.3 \times 10^{-13} \text{ erg cm}^{-2} \text{s}^{-1} \text{Hz} \text{Sr}^{-1}
\]

Now, plug and chug:

\[
T_b = \frac{I_\nu c^2}{2k} = \frac{1.3 \times 10^{-13} \text{ erg cm}^{-2} \text{s}^{-1} \text{Hz} \times (3 \times 10^{10} \text{ cm/s})^2}{2 \times 1.38 \times 10^{-16} \text{ erg/K} \times (10^8 \text{Hz})^2} = 4.2 \times 10^7 \text{ K}
\]

$T_b = 4.2 \times 10^7 \text{ K}$

What energy regime does this correspond to?

Check $\frac{h\nu}{kT}$:

\[
\frac{h\nu}{kT} = \frac{6.626 \times 10^{-27} \text{ erg s} \times 10^8 \text{Hz}}{1.38 \times 10^{-16} \text{ erg/K} \times 4.2 \times 10^7 \text{K}} = 1.1 \times 10^{-10}
\]

$\frac{h\nu}{kT} \ll 1 \Rightarrow \text{R-J regime}$ (big surprise!)

B) The emitting region is actually more compact than indicated by the observed angular diameter. What effect does this have on the value of $T_b$? (2 points)

If $\theta$ is smaller than assumed, then $\Delta \Omega$ decreases, $I_\nu$ increases, $T_b$ increases.

C) At what frequency will this object’s radiation be maximum, if the emission is blackbody? (2 points)

Wien displacement law: $h\nu_{\text{max}} = 2.82 \, k \, T \Rightarrow \nu_{\text{max}} = \frac{2.82 \, k \times 4.2 \times 10^7 \text{K}}{h} = 2.5 \times 10^{18} \text{ Hz}$

D) What can you say about the temperature of the material from the above results? (2 points)

Graybodies: Planck function gives max emission for temp $T$. Therefore we know only that the temp of the object is $\geq T_b$ calculated above.
Problem 3 (8 points)

(Rybicki & Lightman 1.9)

A spherical, opaque object emits as a blackbody at temperature \( T_c \). Surrounding this central object is a spherical shell of material, thermally emitting at a temperature \( T_s (T_s < T_c) \). This shell absorbs in a narrow spectral line; that is, its absorption coefficient becomes large at the frequency \( \nu_0 \) and is negligibly small at other frequencies, such as \( \nu_1 \): \( \alpha_{\nu_0} \gg \alpha_{\nu_1} \) (see Fig. 1.16). The object is observed at frequencies \( \nu_0 \) and \( \nu_1 \) and along two rays \( A \) and \( B \) shown above. Assume that the Planck function does not vary appreciably from \( \nu_0 \) to \( \nu_1 \).

\( A) \quad \text{At which frequency will the observed brightness be larger when observed along ray } A? \text{ Along ray } B? \) (4 points)

Along Ray \( A \):
- Intensity at surface of BB: \( I_\nu = B_\nu(T_c) \)
- Since no (or very little) absorption at \( \nu_1 \), intensity remains \( \sim \)constant along the ray at frequency \( \nu_1 \).
- Now consider \( \nu_0 \).
- Source function in shell (thermal radiation) is \( J_\nu = \frac{\nu^3}{\alpha_\nu} = B_\nu(T_s) \).
- Since \( T_s < T_c \), \( B_\nu(T_s) < B_\nu(T_c) \), and intensity drops as ray passes through shell (\( I_{\nu_0}^A < I_{\nu_1}^A \)).
Along Ray B:
\[ I_{\nu_1}^B = I_{\nu_0}^B = 0 \] (no incident radiation).
Since \( T_s > 0 \), \( B_\nu(T_s) > B_\nu(0) \), and intensity increases as ray passes through shell \( (I_{\nu_0}^B > I_{\nu_1}^B) \).

B) Answer the preceding questions if \( T_s > T_c \) (4 points)

Along Ray A:
Since \( T_s > T_c \), \( B_\nu(T_s) > B_\nu(T_c) \), and intensity increases as ray passes through shell \( (I_{\nu_0}^A > I_{\nu_1}^A) \).

Along Ray B:
Since \( T_s > 0 \) (still), \( B_\nu(T_s) > B_\nu(0) \), and intensity still increases as ray passes through shell \( (I_{\nu_0}^B > I_{\nu_1}^B) \).

Illustration of all four cases:

![Intensity from a blackbody surrounded by a thermal absorbing shell](image)

Problem 4 (8 points)

(Courtesy J. Moran)
Calculate the quietest (i.e., darkest) place in the radio spectrum. Neglect noise from the earth’s atmosphere. At low frequencies the sky noise is dominated by synchrotron emission from relativistic electrons rattling around all over the galaxy. Away from the galaxy plane, the brightness temperature is
\[ T_B \simeq 180 K \left( \frac{\nu}{180 MHz} \right)^{-2.6} \]
which is more or less independent of direction. At high frequencies, the CMB dominates with \( T_B \sim 2.7 K \) in all directions.
A) What is the frequency of minimum background brightness temperature? Of minimum background flux density? (4 points)

Background brightness temperature: 
\[ T_B(\nu) = 180 \text{ K} \left( \frac{\nu}{180 \text{ MHz}} \right)^{-2.6} + 2.7 \text{ K.} \]

No minimum \( T_B \); asymptotically approaches 2.7 K.

Minimum flux density:

\[ S_\nu = I_\nu \Omega = \frac{2kT_B}{c^2} \nu^2 \Omega \] (small angles)
\[ = \frac{2k}{c^2} \left[ 180 \text{ K}(180 \text{ MHz})^{2.6} \nu^{-0.6} + 2.7 \text{ K} \nu^2 \right] \]
\[ \frac{\partial S_\nu}{\partial \nu} = 0 \text{ at the minimum} \]
\[ = \frac{2k}{c^2} \left[ 180 \text{ K}(180 \text{ MHz})^{2.6}(-0.6)\nu_{\text{min}}^{-1.6} + 5.4 \text{ K} \nu_{\text{min}} \right] \]

\[ \nu_{\text{min}} = 570 \text{ MHz} \]

B) What is the incident power on the earth from 10 MHz → 1 THz (10⁶ – 10¹² Hz)? If we intercept this power could we reduce our reliance on fossil fuels? (4 points)

\[ I_\nu = \frac{2k}{c^2} \left[ 180 \text{ K}(180 \text{ MHz})^{2.6} \nu^{-0.6} + 2.7 \text{ K} \nu^2 \right] \]
\[ S_\nu = \int_0^{2\pi} \int_0^{\pi/2} I_\nu \cos \theta \sin \theta d\theta d\phi \]
where $S_\nu$ is integrated over the total amount of sky visible to a patch of earth’s surface:

Assume that $I_\nu$ is uniform and isotropic...

$$F = \left[ \int_1^{10^{12} \text{Hz}} I_\nu \, d\nu \right] \left[ \int_0^{2\pi} \int_0^{\pi/2} \cos \theta \sin \theta \, d\theta \, d\phi \right]$$

$$= \left[ \int_1^{10^{12} \text{Hz}} \frac{2k}{c^2} [180K(1.8 \times 10^8 \text{Hz})^{2.6} \nu^{-0.6} + 2.7K\nu^2] \, d\nu \right] \left[ 2\pi \times \frac{1}{2} (\sin^2 \theta |_{\theta = \pi/2}) = \pi \text{ Sr} \right]$$

$$= 0.28\pi \text{ erg cm}^{-2} \text{ s} \text{ Sr}$$

So flux per unit area is $0.28\pi \text{ erg cm}^{-2} \text{ s} \text{ Sr}$, and multiplying by the surface area of the earth ($4\pi R_\oplus^2$) gives the total power:

$$P = (4\pi R_\oplus^2)(0.28\pi) = 4.5 \times 10^{18} \text{ erg/s} \text{ or } \sim 5 \times 10^{11} \text{ W}.$$  

Compare this with the power of the sun, $P_\odot \sim 10^{17} \text{ W}$, and the energy used to generate electricity from fossil fuels worldwide, currently around, $P_f \sim 10^{13} \text{ W}$.

Problem 5 (8 points)

(Courtesy J. Moran)

The full moon at millimeter wavelengths has a brightness temperature distribution that might be approximated as (see figure): $T_B(\theta) = T_0 + T_1 \cos \theta$.

A) What is the flux density at the earth? Hint: First calculate the brightness temperature as a function of polar angle in the earth coordinate system. (4 points)

$$T_B(\theta) = T_0 + T_1 \cos \theta$$

Geometry: $\cos \theta = \frac{\sqrt{R^2 - x^2}}{R}$, and $\phi = x/D \Rightarrow x = D\phi$

$$S_\nu = \int I_\nu \cos \theta \, d\Omega \text{ (In earth-based coordinates, } \theta \rightarrow \phi).$$

$$I_\nu = \frac{2kT_B\nu^2}{c^2}$$

Earth-based coordinates: $T_B(\phi) = T_0 + T_1 \frac{\sqrt{R^2 - D^2 \phi^2}}{R}$. Then:
\[ S_\nu = \frac{2k\nu^2}{c^2} \int_0^{R/D} (T_0 + T_1 \sqrt{\frac{R^2-D^2 \phi^2}{R}}) \cos \phi \sin \phi d\phi \times 2\pi \]

\[ \cos \phi \sim 1 \text{ and } \sin \phi \sim \phi \text{ (small angles)} \]

\[ = \frac{4\pi k\nu^2}{c^2} \left[ \int_0^{R/D} T_0 \phi d\phi + \int_0^{R/D} \phi T_1 \frac{D}{R} \sqrt{(R/D)^2 - \phi^2} d\phi \right] \]

\[ = \frac{4\pi k\nu^2}{c^2} \left( \frac{1}{2} T_0 (\frac{R}{D})^2 - \frac{1}{2} T_1 \frac{D}{R} \int u^{1/2} du \right) \text{ Note: } u = (\frac{R}{D})^2 - \phi^2 \]

\[ = \frac{2\pi k\nu^2}{c^2} \left( T_0 (\frac{R}{D})^2 - T_1 \frac{D}{R} \frac{2}{3} \left( 0 - (\frac{R}{D})^3 \right) \right) \]

\[ S_\nu = \frac{2\pi k\nu^2}{c^2} \left( \frac{R}{D} \right)^2 (T_0 + \frac{2}{3} T_1) \]

**B) What is the mean brightness temperature, i.e., "disk temperature"? (4 points)**

Mean \( T_D \) of the moon:

\[ T_{mean} = \frac{1}{\Omega_{moon}} \int_0^{R/D} (T_0 + T_1 \sqrt{\frac{R^2-D^2 \phi^2}{R}}) 2\pi \phi d\phi \]

where \( \Omega_{moon} = \pi (\frac{R}{D})^2 \), and we have already evaluated the integral:

\[ T_{mean} = \frac{1}{\pi (\frac{D}{R})^2} \left[ R^2 (T_0 + \frac{2}{3} T_1) \pi \right] \]

\[ = T_0 + \frac{2}{3} T_1 \]